

Generalized Pareto Curves: Theory and Applications

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The Pareto distribution

- The distribution of income and wealth is well approximated by the Pareto (1896) distribution.
- Minimum income $x_0 > 0$.
- Linear relationship between $\log(\text{rank})$ and $\log(\text{income})$:

$$\forall x \geq x_0 \quad \log \mathbb{P}\{X > x\} = -\alpha \log(x/x_0)$$

- Hence the “power law”:

$$\forall x \geq x_0 \quad \mathbb{P}\{X > x\} = (x/x_0)^{-\alpha}$$

- $\alpha =$ Pareto coefficient

Beyond Pareto (I)

Pareto's functional form is now an accepted stylized fact, at least for the top.

- ⇒ We introduce **generalized Pareto curves** to characterize power law behavior in a flexible way. We notice recurrent and economically meaningful deviations from the strict Pareto distribution.

Beyond Pareto (II)

Wide use in empirical work, in particular to deal with censoring of tax data (Kuznet, 1953; Feenberg and Poterba, 1992; Piketty, 2001; Piketty and Saez, 2003; etc.)

⇒ We develop an empirical method to use tax data that is more precise and preserves deviations from strict Paretian behavior: **generalized Pareto interpolation.**

Beyond Pareto (III)

Literature trying to explain this functional form (Champernowne, 1953; Piketty and Zucman, 2014; Gabaix et al., 2016; Benhabib and Bisin, 2016; etc.)

⇒ We show how small modifications to the existing literature can account for the new empirical findings, and potentially help identifying key parameters of the income/wealth generating process.

Generalized Pareto Curves

Generalized Pareto curves

- Assume a Pareto distribution with coefficient α .
- Quantile function $Q : p \mapsto x_0(1 - p)^{-1/\alpha}$.
- Then:

$$\frac{\mathbb{E}[X|X > Q(p)]}{Q(p)} = \frac{\alpha}{\alpha - 1} = b$$

is constant.

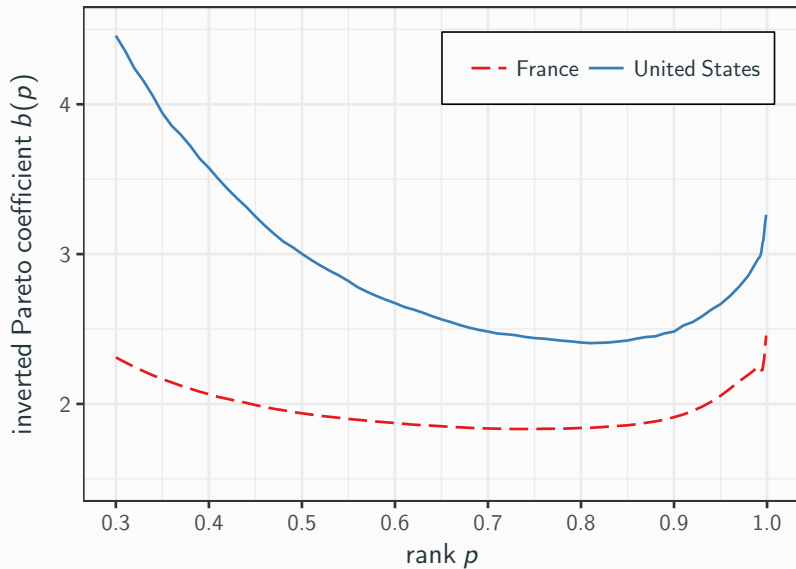
- b is the **inverted Pareto coefficient**.
- More generally, we can define:

$$b(p) = \frac{\mathbb{E}[X|X > Q(p)]}{Q(p)}$$

- $p \mapsto b(p)$ is the **generalized Pareto curve**.

Pre-tax national income

Year 2010

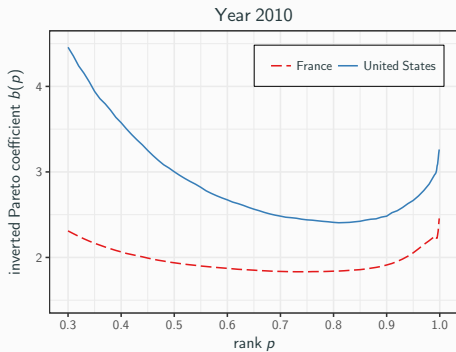
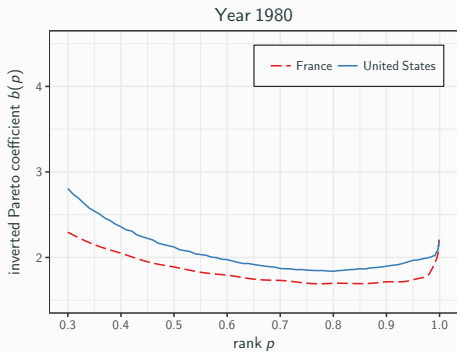


Power laws: a more flexible definition

- More flexible definition of power laws. ▶ definition
 - (Based on Karamata's (1930) theory of regular variations.)
- Two types of distributions, excluding pathological cases:

Power laws	Thin tails
Pareto	Normal
Student's t	Log-normal
...	...
$\lim_{p \rightarrow 1} b(p) > 1$	$\lim_{p \rightarrow 1} b(p) = 1$

The evolution of pre-tax national income



Generalized Pareto Interpolation

An example of tabulated data

Tax data is typically available as:

Income bracket	Bracket size	Bracket average income
From 0 to 1000	300 000	500
From 1000 to 10 000	600 000	5 000
From 10 000 to 50 000	80 000	30 000
More than 50 000	20 000	200 000

- We know $Q(p)$ and $b(p)$ for a few values of p .
- How can we reconstruct an entire distribution based on that information?

The interpolation problem

- **Problem:** the interpolation must be constrained so that $Q(p)$ and $b(p)$ are consistent with the data and with each other.
- With $x = -\log(1 - p)$, define the function:

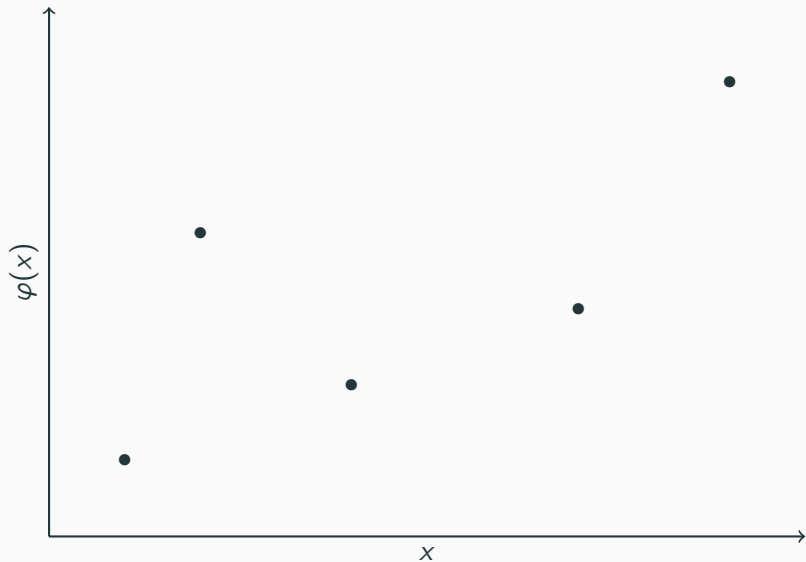
$$\varphi(x) = -\log \int_{1-e^{-x}}^1 Q(u) du$$

- Its derivative is directly linked to $b(p)$:

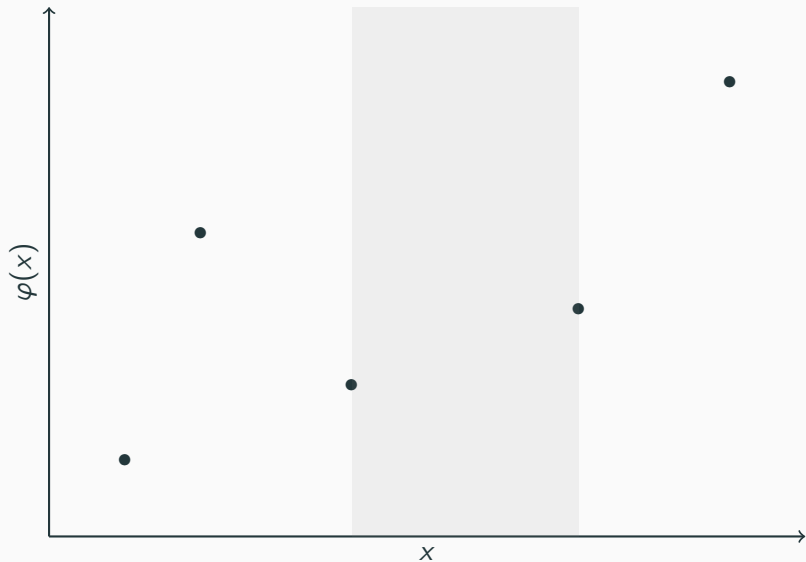
$$\varphi'(x) = 1/b(p)$$

- The initial problem is equivalent to interpolating the function φ , knowing, at the interpolation points:
 - its **value**
 - its **first derivative**

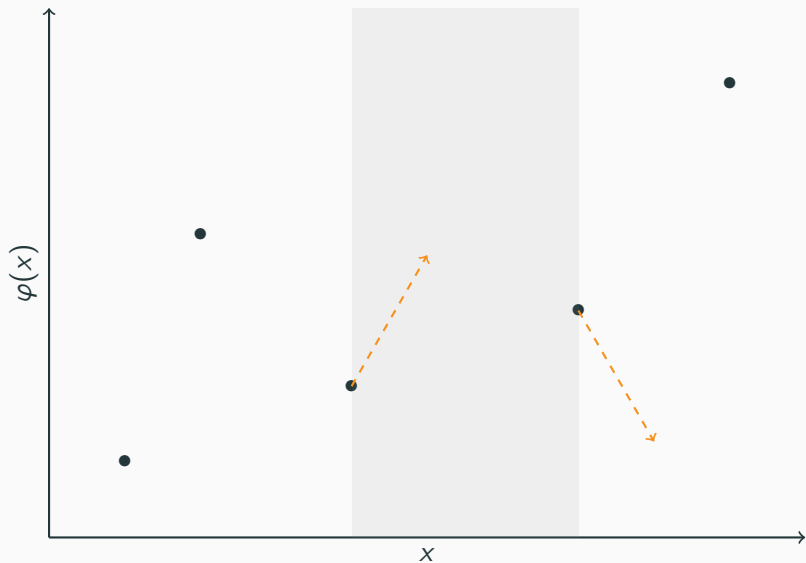
Cubic spline interpolation



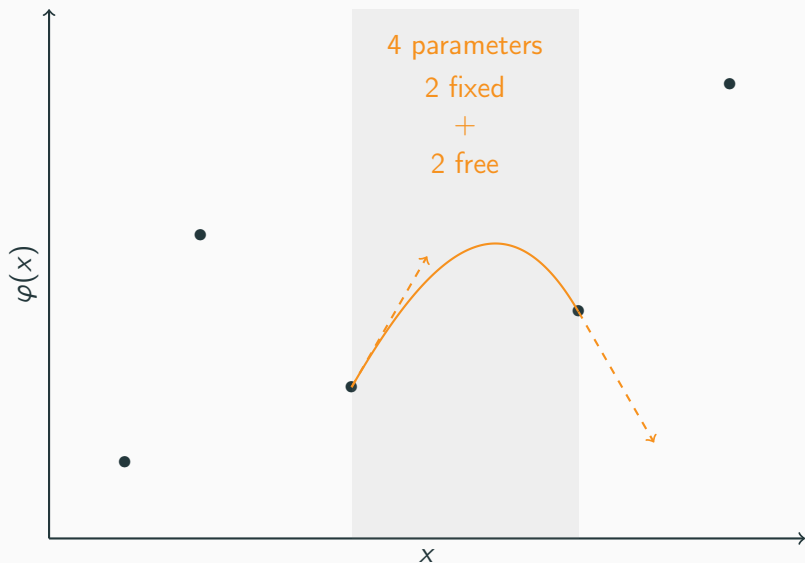
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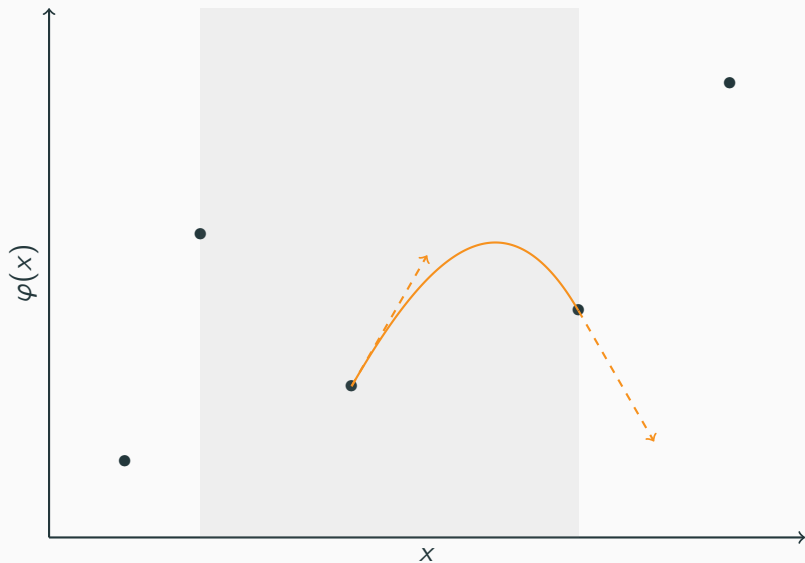
Cubic spline interpolation



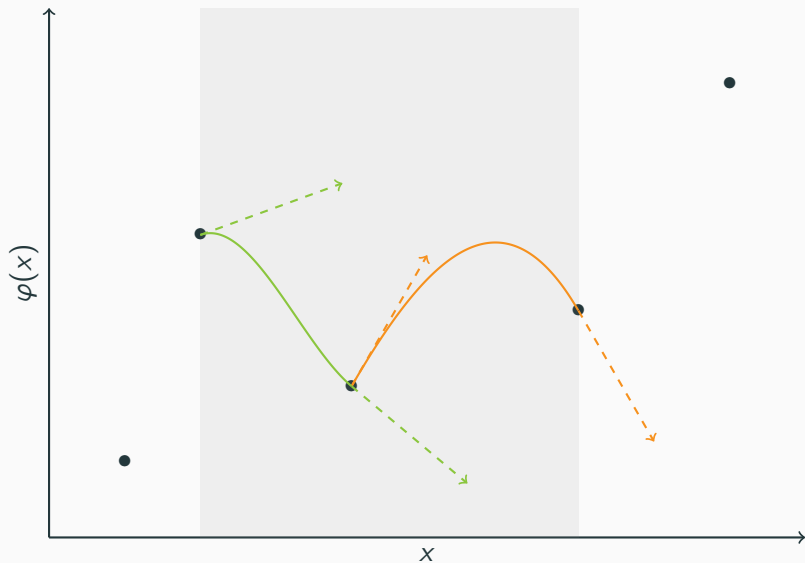
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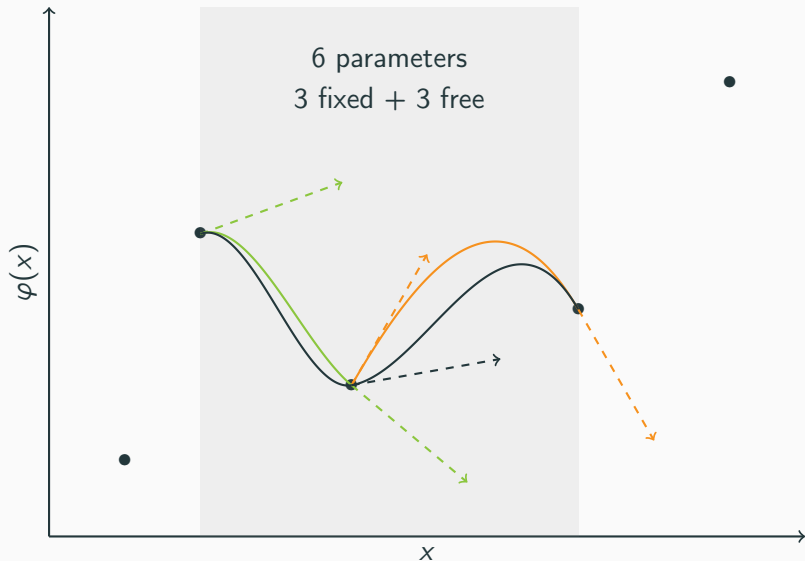
Cubic spline interpolation



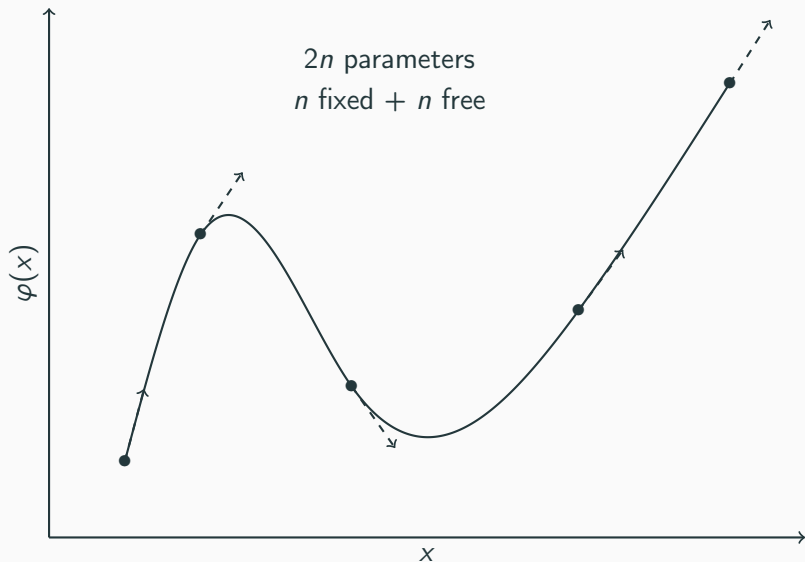
Cubic spline interpolation



Cubic spline interpolation



Cubic spline interpolation



Cubic spline interpolation

We determine the n free parameters by looking for the “most regular” curve. There are two **equivalent** approaches:

- Enforce continuity of **second** derivative at the jointures (+ two boundary conditions).
- Minimize the **curvature** of the overall curve:

$$\min \int_{x_1}^{x_K} \{\varphi''(x)\}^2 dx$$

Requires solving a linear system of equations.

Quintic splines

- We **already know** the first derivative.
- Cubic spline? Not smooth enough.
- Quintic spline:
 - n extra degrees of freedom.
 - one extra level of differentiability.
 - same principle as cubic spline, but applied to the derivative.

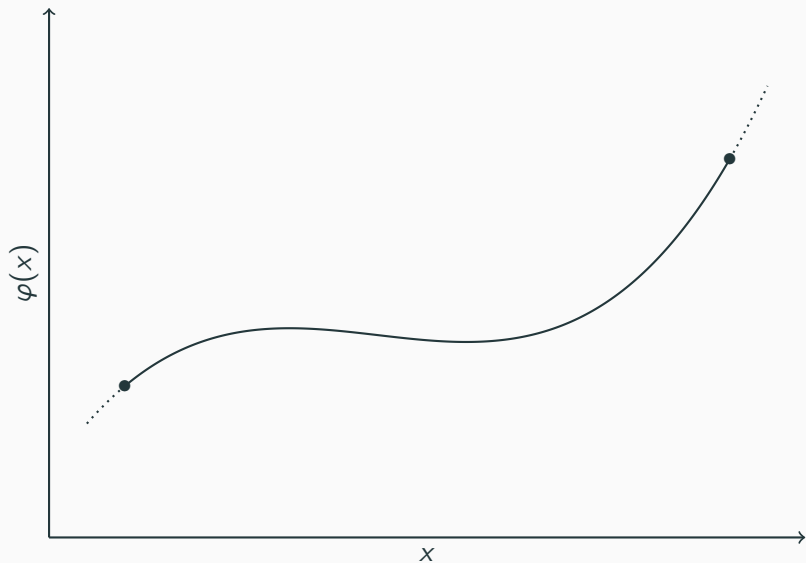
Making sure the quantiles are increasing

- No guarantee yet that the quantile function is increasing. (Although most of the time it is.)
- Quantile function increasing if and only if:

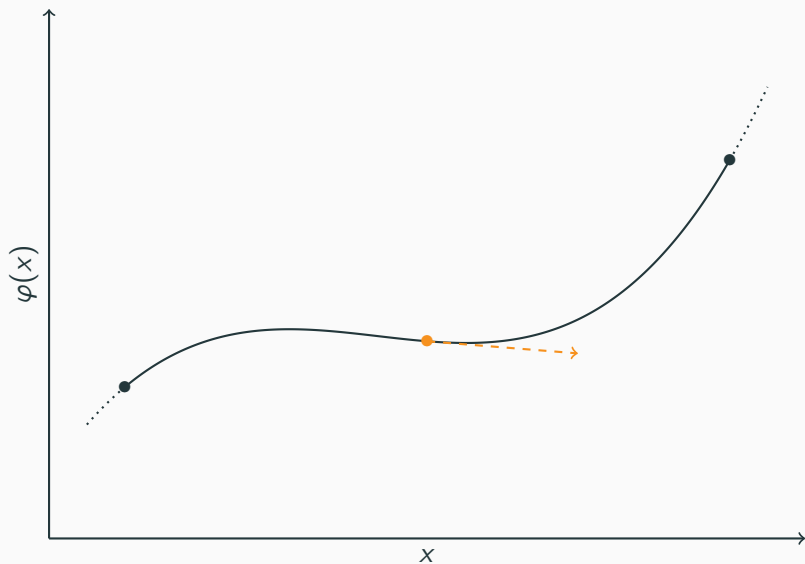
$$\varphi''(x) + \varphi'(x)(1 - \varphi'(x)) > 0$$

- Derive fairly general sufficient conditions on the parameters of the splines (based on their Bernstein representation). ▶ conditions

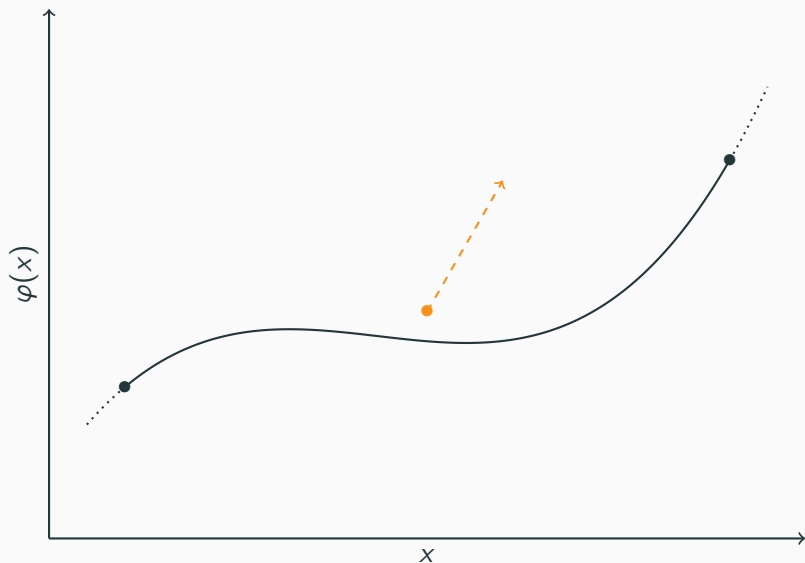
Making sure the quantiles are increasing



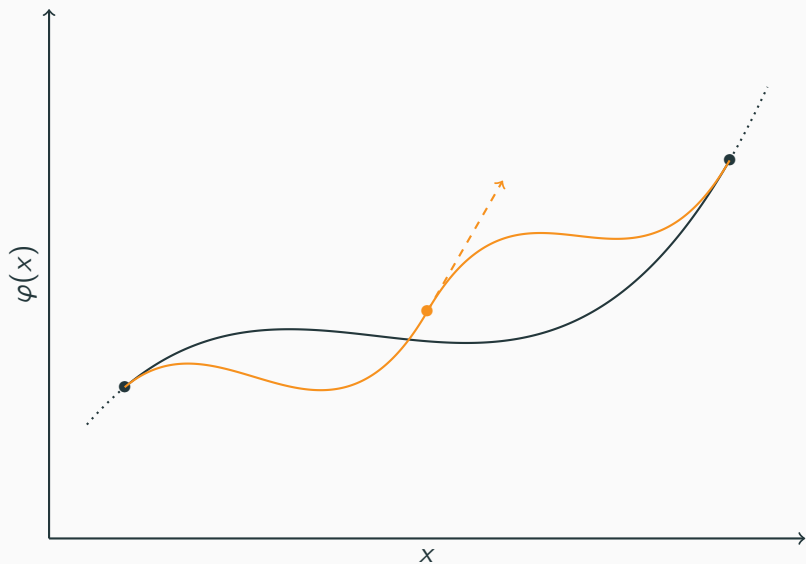
Making sure the quantiles are increasing



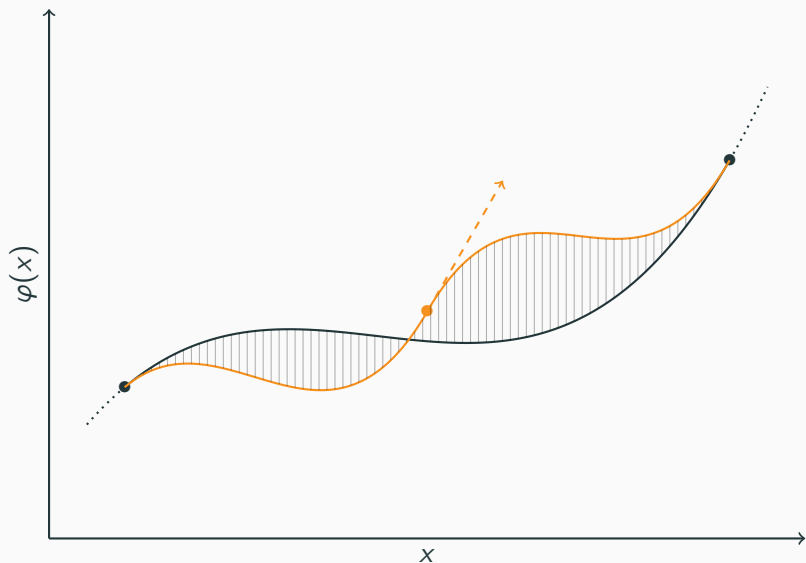
Making sure the quantiles are increasing



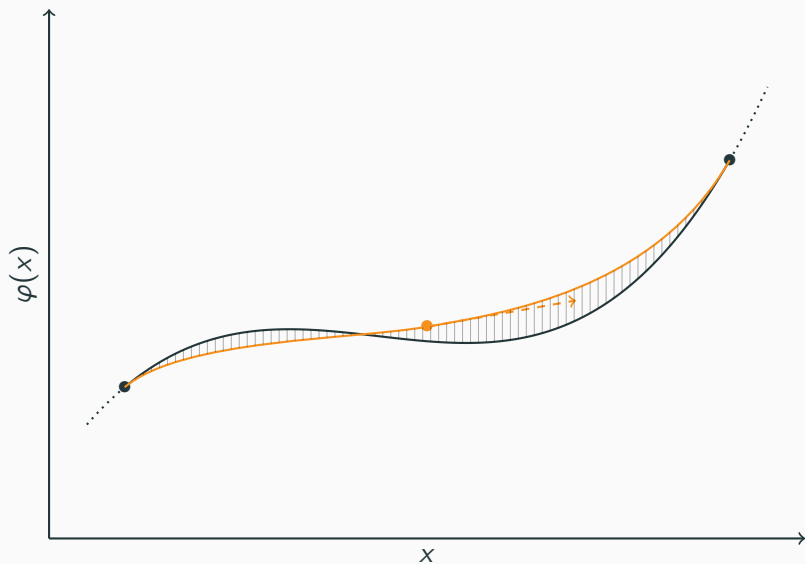
Making sure the quantiles are increasing



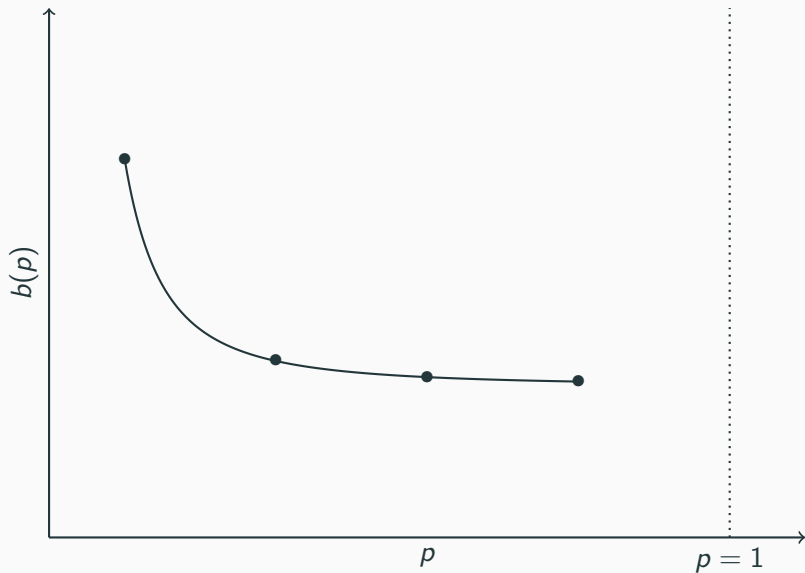
Making sure the quantiles are increasing



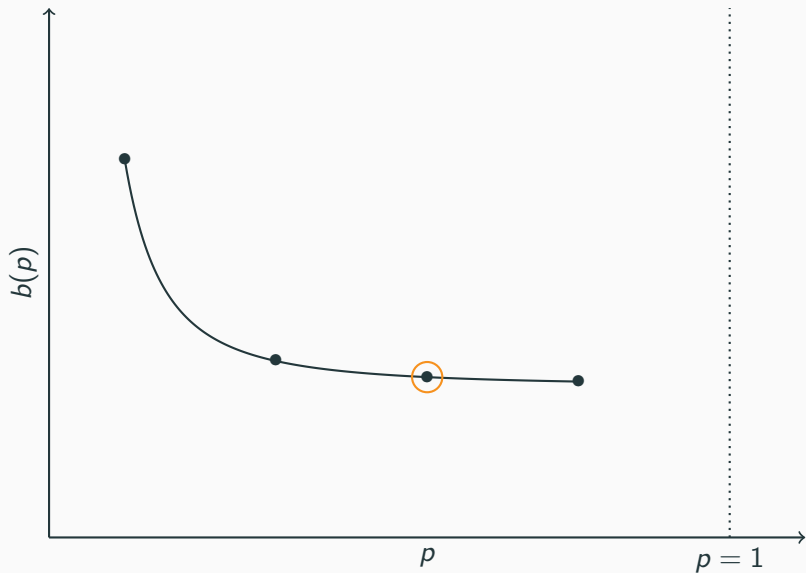
Making sure the quantiles are increasing



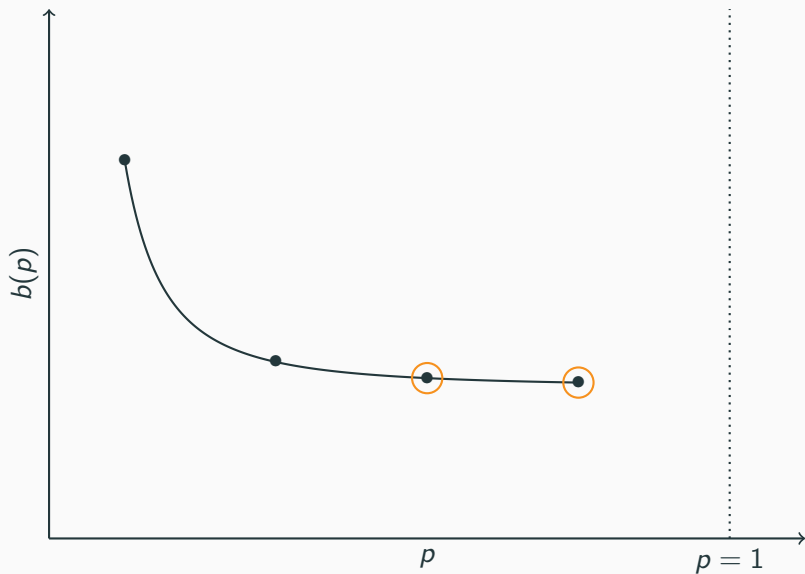
Extrapolation beyond the last threshold



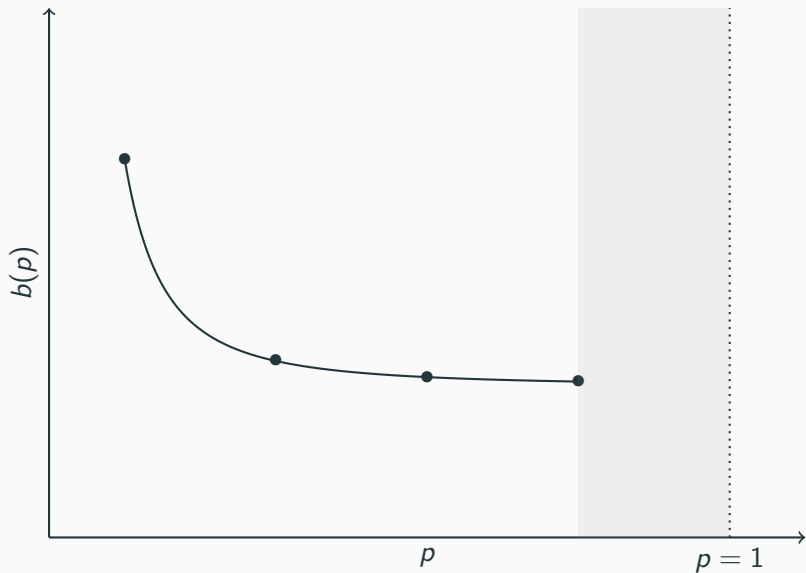
Extrapolation beyond the last threshold



Extrapolation beyond the last threshold



Extrapolation beyond the last threshold



Extrapolation beyond the last threshold

- Extrapolation using the **generalized Pareto distribution**.
- “Universal” model of distribution tails (see extreme value theory).
- 3 parameters, identified using:
 - last threshold.
 - last inverted Pareto coefficient $b(p)$.
 - differentiability condition at the jointure.

Testing the method

Alternative method #1

Method M1: piecewise constant $b(p)$

- assumes $b(p)$ constant within each bracket.
- not entirely consistent with the input data.
- not always self-consistent.

See Piketty (2001), Piketty and Saez (2003).

Alternative method #2

Method M2: piecewise Pareto distribution

- uses $\log(1 - F(x)) = A - B \log(x)$ within each bracket.
- only uses threshold information, not shares.

See Kuznet (1953), Feenberg and Poterba (1992).

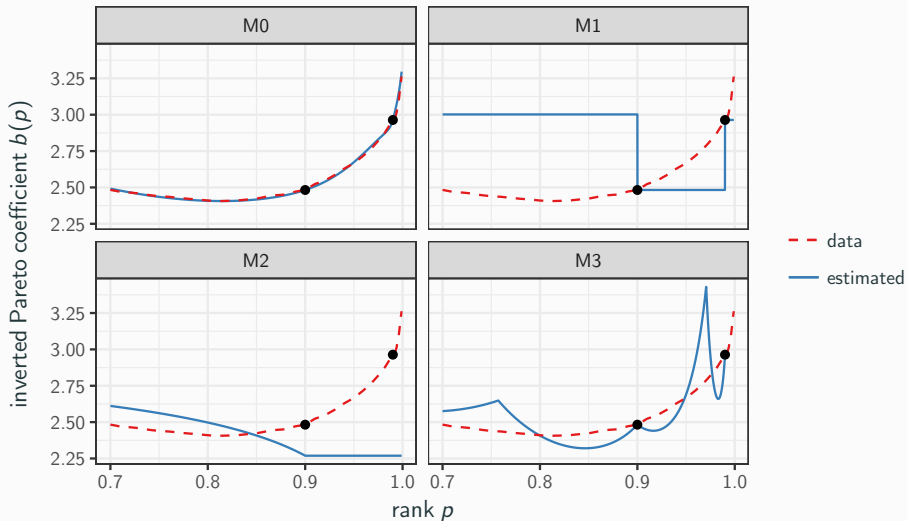
Alternative method #3

Method M3: mean-split histogram

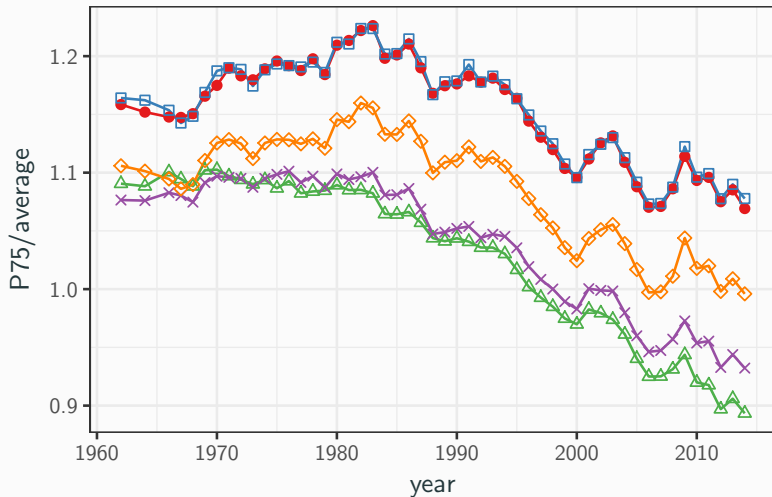
- divide each bracket in two parts.
- defines a uniform distribution on each part.
- the breakpoint is the mean income inside the bracket.

- Data from France and the United States coming from exhaustive or quasi-exhaustive microdata:
 - France: Garbinti, Goupille-Lebret and Piketty (2016).
 - United States: Piketty, Saez and Zucman (2016).
- Create a tabulation $p = 10\%, 50\%, 90\%, 99\%$.
- Compare estimated and actual value.

US pre-tax national income, 2010: generalized Pareto curve

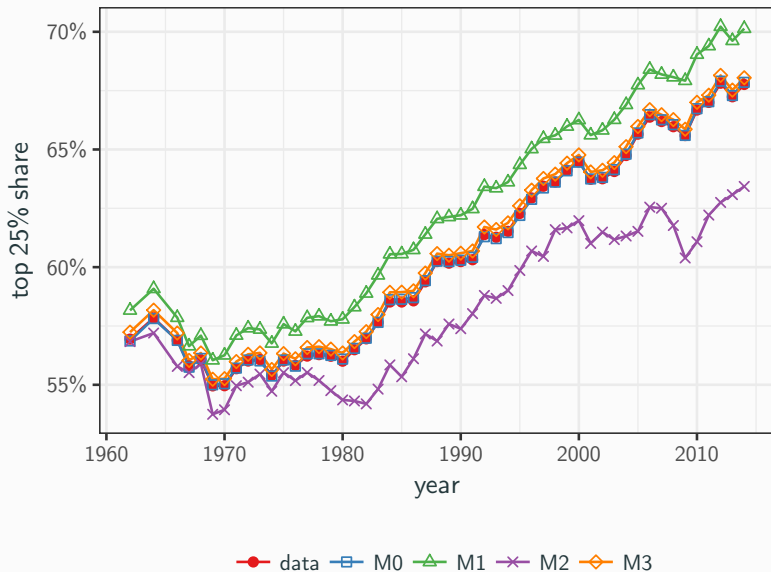


US pre-tax national income, P75/average

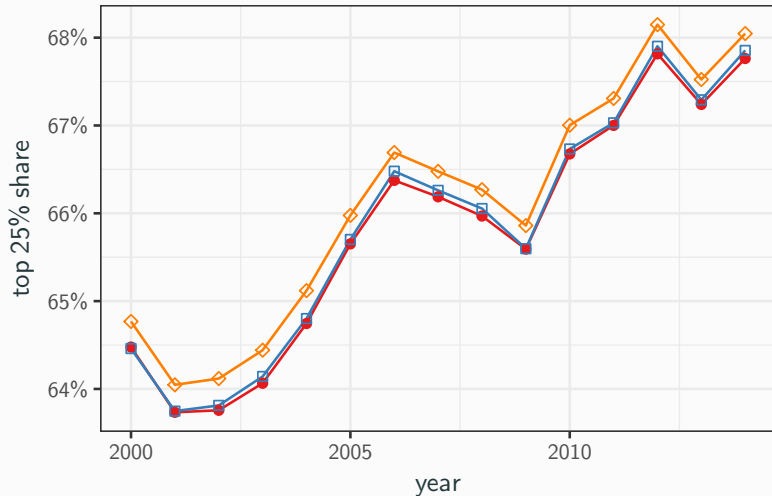


—●— data —■— M0 —▲— M1 —×— M2 —◇— M3

US pre-tax national income, top 25% share



US pre-tax national income, top 25% share (2000–2014)



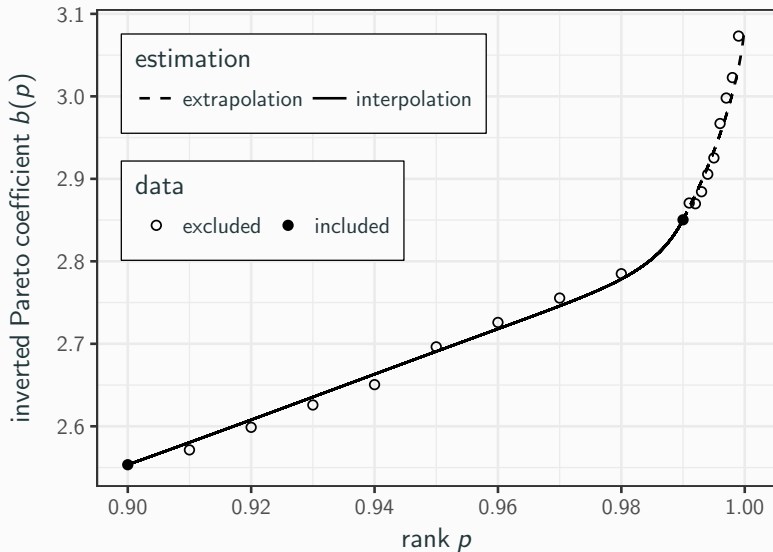
—●— data —■— M0 —▲— M1 —×— M2 —◇— M3

Comparison table

		mean percentage gap between estimated and observed values			
		M0	M1	M2	M3
United States (1962–2014)	Top 70% share	0.059% (ref.)	2.3% (×38)	6.4% (×109)	0.054% (×0.92)
	Top 25% share	0.093% (ref.)	3% (×32)	3.8% (×41)	0.54% (×5.8)
	Top 5% share	0.058% (ref.)	0.84% (×14)	4.4% (×76)	0.83% (×14)
	P30/average	0.43% (ref.)	55% (×125)	29% (×67)	1.4% (×3.3)
	P75/average	0.32% (ref.)	11% (×35)	9.9% (×31)	5.8% (×18)
	P95/average	0.3% (ref.)	4.4% (×15)	3.6% (×12)	1.3% (×4.5)

Extrapolation: generalized Pareto curve

United States, 2014



Extrapolation: comparison table

Estimating the top 1% from the top 10% and the top 5%

		mean percentage gap between estimated and observed values		
		M0	M1	M2
United States (1962–2014)	Top 1% share	0.78% (ref.)	5.2% (×6.7)	40% (×52)
	P99/average	1.8% (ref.)	8.4% (×4.7)	13% (×7.2)

Compare with a survey

Take the example of the US distribution of pre-tax national income.
What precision can we expect using subsamples of the data?

mean percentage gap between estimated and observed values for a survey
with simple random sampling and sample size n out of 10^8

	$n = 10^3$	$n = 10^4$	$n = 10^5$	$n = 10^6$	$n = 10^7$	$n = 10^8$
Top 5% share	13.40%	6.68%	3.34%	1.34%	0.44%	0%
Top 1% share	27.54%	14.51%	7.39%	2.98%	0.97%	0%
Top 0.1% share	51.25%	33.08%	17.89%	7.41%	2.43%	0%

Explaining the shape of wealth and income distributions

The basics

- Pareto distributions emerge through **proportional random growth** with frictions.
- Champernowne (1953), Simon (1955), Whittle (1957), Piketty and Zucman (2015), Gabaix et al. (2016), etc.
- Example: Kesten (1973) process.

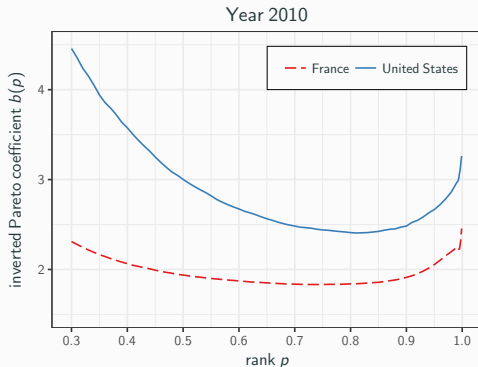
$$X_t = \omega_t X_{t-1} + \varepsilon_t$$

- Power-law tail: $1 - F(x) \sim Cx^{-\alpha}$ with

$$\mathbb{E}[|\omega_t|^\alpha] = 1$$

Limitations

It is hard to explain with those models why $b(p)$ increases toward the top of the distribution.



Need to make the variance and/or mean of multiplicative shocks vary with income.

Generalized model

Two main results:

- If the mean and variance of multiplicative shocks **converges to a constant**, then we get a power law in the sense that $\lim_{p \rightarrow 1} b(p) > 1$.
- The U-shaped profile of $b(p)$ can be explained by a U-shaped profile of variance.

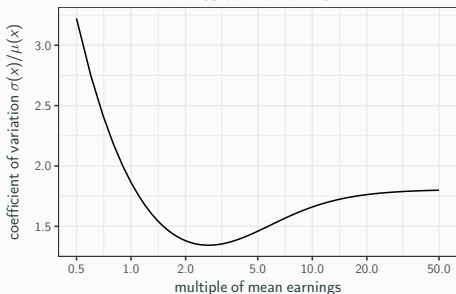
Connection to existing literature

- Empirical literature noticed a U-shaped profile of the variance of income shocks (Chauvel and Hartung, 2014; Hardy and Ziliak, 2014; Bania and Leete, 2009; Guvenen et al., 2015).
- Gabaix et al. (2016) emphasized the need to introduce scale dependence to explain the speed of increase of inequality.

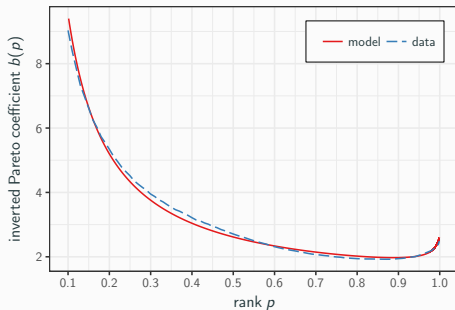
Simple calibration

US labor income, 2014. We assume constant mean, changing variance.

Volatility of earnings growth
calibrated to match the distribution
of US labor income in 2014



Generalized Pareto curve
United States labor income in 2014



Summing up

Three main points:

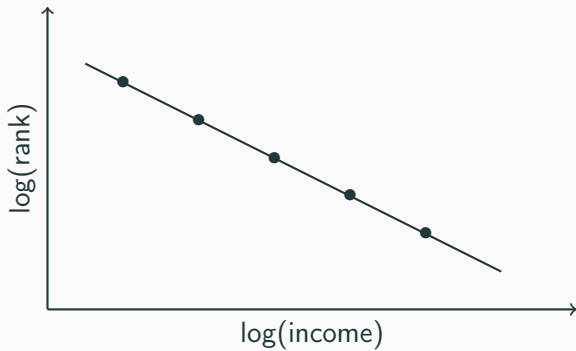
- we can use generalized Pareto curves $b(p)$ to explore power laws in an extended sense, and we observe significant deviations from Pareto behavior.
- we can increase the precision of empirical work by taking these deviations into account.
- we can extend existing models of the income and wealth distribution to account for the new findings.



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Additional slides

The Pareto distribution



Power laws: Karamata's (1930) definition

- For some α , write $1 - F(x)$ as:

$$1 - F(x) = \mathbb{P}\{X > x\} = L(x)x^{-\alpha}$$

- X is an **asymptotic power law** if L is **slowly varying**:

$$\forall \lambda > 0 \quad \lim_{x \rightarrow +\infty} \frac{L(\lambda x)}{L(x)} = 1$$

- Includes cases where $L(x) \rightarrow \text{constant}$, but also (say)
 $L(x) = (\log x)^\beta$, $\beta \in \mathbb{R}$.

Thin tails: rapidly varying functions

- Otherwise, $1 - F$ may be **rapidly varying**, meaning:

$$\forall \lambda > 1 \quad \lim_{x \rightarrow +\infty} \frac{1 - F(\lambda x)}{1 - F(x)} = 0$$

- That corresponds to **thin tailed** distributions:
 - Normal
 - Log-normal
 - Exponential
 - ...

A simple typology of distributions

Category	Examples	$b(p)$ behavior
Power laws	Pareto Student's t Dagum	$\lim_{p \rightarrow 1} b(p) > 1$
Thin tails	Normal Log-normal Exponential	$\lim_{p \rightarrow 1} b(p) = 1$
Pathological cases	none	oscillates indefinitely (no convergence)

Sufficient conditions for an increasing quantile function

Cargo and Shisha (1966)

Let $P(x) = c_0 + c_1x_1 + \cdots + c_nx^n$ be a polynomial of degree $n \geq 0$ with real coefficients. Then:

$$\forall x \in [0, 1] \quad \min_{0 \leq i \leq n} b_i \leq P(x) \leq \max_{0 \leq i \leq n} b_i$$

where:

$$b_i = \sum_{r=0}^n c_r \binom{i}{r} / \binom{n}{r}$$

Which error?

- Two possible definitions of “the error”
 - the error with respect to the **actual population value**.
 - the error with respect to the **underlying statistical model**.
- The first error is more relevant for causal inference.
- We consider the second kind of error.

Two components

- What if the population was infinite (no sampling variability)?
 - There would still be an error because the actual distribution doesn't match our interpolation exactly.
 - **misspecification error**
- What if the actual distribution matched the functional forms we use to interpolate?
 - There would still be an error due to sampling variability.
 - **sampling error**
- The total error is the sum of both.
- We'll focus on the unconstrained estimation (explicit results, still covers the overwhelming majority of cases).

Sampling error

- Finite variance case:
 - Standard approach (CLT-type results + delta method).
 - Asymptotic normality.
- Infinite variance case:
 - Generalized CLT (Gnedenko and Kolmogorov, 1968).
 - Converge to a stable distribution.
- In both cases: sampling error negligible.

Misspecification error: theory

- Explicit expression for this error (Peano kernel theorem).
- Depends on two elements:
 - the interpolation percentiles.
 - φ'''
- Think of φ''' as a residual. It captures all the features of the distribution not taken into account by the interpolation method.

Misspecification error: applications

- We estimate φ''' when we have access to microdata.
- We can plug-in these estimates in the error formula to:
 - get bounds on the error in the general case.
 - solve the inverse problem: how to place thresholds optimally?

Optimal position of thresholds

	3 brackets	4 brackets	5 brackets	6 brackets	7 brackets
	10.0%	10.0%	10.0%	10.0%	10.0%
	68.7%	53.4%	43.0%	36.8%	32.6%
	95.2%	83.4%	70.4%	60.7%	53.3%
optimal placement of thresholds	99.9%	97.1%	89.3%	80.2%	71.8%
		99.9%	98.0%	93.1%	86.2%
			99.9%	98.6%	95.4%
				99.9%	98.9%
					99.9%
maximum relative error on top shares	0.91%	0.32%	0.14%	0.08%	0.05%